GPS user position solution

GPS range measurements are termed “pseudo-ranges” because they contain clock bias errors. The geometric range \( R \) from point \((x, y, z)\) to a satellite at \((x_s, y_s, z_s)\) is

\[
R(x, y, z) = \sqrt{(x - x_s)^2 + (y - y_s)^2 + (z - z_s)^2}
\]  

[1]

But the pseudo-range as measured by a receiver at point \((x, y, z)\) with clock bias \(t_b\) is

\[
\rho(x, y, z, t_b) = R(x, y, z) + ct_b
\]  

[2]

Ignoring second and higher order terms, this can be approximated and linearised by a multi-variable, generalised Taylor Series expansion

\[
\rho(x + \Delta x, y + \Delta y, z + \Delta z, t + \Delta t) \approx \rho(x, y, z, t) + \frac{\partial \rho}{\partial x} \Delta x + \frac{\partial \rho}{\partial y} \Delta y + \frac{\partial \rho}{\partial z} \Delta z + \frac{\partial \rho}{\partial t} \Delta t
\]  

[3]

Let \((x, y, z)\) and time \(t_b\) be estimates of receiver position and clock bias; and let \(\Delta x, \Delta y, \Delta z\) and \(\Delta t_b\) be errors in these estimates. Better estimates are

\[
x' = x + \Delta x \quad y' = y + \Delta y \quad z' = z + \Delta z \quad t'_b = t_b + \Delta t_b
\]

And equation 3 can be re-written

\[
\rho(x', y', z', t'_b) = \rho(x, y, z, t_b) + \frac{\partial \rho}{\partial x} \Delta x + \frac{\partial \rho}{\partial y} \Delta y + \frac{\partial \rho}{\partial z} \Delta z + \frac{\partial \rho}{\partial t} \Delta t_b
\]  

[4]

Abbreviating \(R(x, y, z)\) to \(R\), the partial derivatives of \(\rho\) are

\[
\frac{\partial \rho}{\partial x} = \frac{x - x_s}{R} \quad \frac{\partial \rho}{\partial y} = \frac{y - y_s}{R} \quad \frac{\partial \rho}{\partial z} = \frac{z - z_s}{R} \quad \frac{\partial \rho}{\partial t} = c
\]

Substituting into equation 4 gives

\[
\rho(x', y', z', t'_b) = \rho(x, y, z, t_b) + \frac{x - x_s}{R} \Delta x + \frac{y - y_s}{R} \Delta y + \frac{z - z_s}{R} \Delta z + c \Delta t_b
\]  

[5]

The (notionally more accurate) left-hand-side of equation 5 is now replaced with an actual pseudo-range measurement made by the GPS receiver

\[
ct_{RX} - ct_{TX} = \rho(x, y, z, t_b) + \frac{x - x_s}{R} \Delta x + \frac{y - y_s}{R} \Delta y + \frac{z - z_s}{R} \Delta z + c \Delta t_b
\]  

[6]

This is a linear equation in four unknowns: \(\Delta x, \Delta y, \Delta z\), and \(\Delta t_b\).
Let error term $\Delta \rho$ be the difference between measured and estimated pseudo-range

$$\Delta \rho = ct_{rx} - ct_{tx} - \rho(x, y, z, t_b)$$

And re-write equation 6 in terms of $\Delta \rho$

$$\Delta \rho_i = \frac{x_i - x}{R_x} \Delta x + \frac{y_i - y}{R_y} \Delta y + \frac{z_i - z}{R_z} \Delta z + c \Delta t_b$$

[7]

At least four range measurements are needed to solve for four unknowns. The simultaneous equations are conveniently written in matrix form

$$\begin{bmatrix} \Delta \rho_1 \\ \Delta \rho_2 \\ \Delta \rho_3 \\ \Delta \rho_4 \end{bmatrix} = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{bmatrix} \begin{bmatrix} c \\ c \\ c \\ c \end{bmatrix}$$

$$\begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \\ \Delta t_b \end{bmatrix}$$

[8]

Equation 8 can be expressed as

$$\Delta \rho = H \cdot \Delta u$$

[9]

In the four-satellite case, we simply multiply the pseudo-range errors by the matrix inverse of $H$ to find the four unknowns

$$\Delta u = H^{-1} \cdot \Delta \rho$$

[10]

To solve more than four equations, we take the generalised or left inverse of $H$

$$\Delta u = (H^T \cdot H)^{-1} \cdot H^T \cdot \Delta \rho$$

[11]

$H^T \cdot H$ is always 4x4, so we still only need to compute a 4x4 inverse.

The influence of each satellite can be adjusted using weighted least squares

$$\Delta u = (H^T \cdot W \cdot H)^{-1} \cdot H^T \cdot W \cdot \Delta \rho$$

[12]

The process is repeated until error magnitude $|\Delta x \Delta y \Delta z |$ is small. Typically, only five or six iterations are required. Zero starting values can be used for $x, y, z$ and $t_b$.

Satellite positions should be transformed from ECEF to ECI co-ordinates inside the loop, since better estimates of signal flight times are available after each iteration.