GPS user position solution

GPS range measurements are termed "pseudo-ranges" because they contain clock bias errors. The geometric range R from point *(x, y, z)* to a satellite at (x_s, y_s, z_s) is

$$R(x, y, z) = \sqrt{(x - x_s)^2 + (y - y_s)^2 + (z - z_s)^2} \qquad [1]$$

But the pseudo-range as measured by a receiver at point *(x, y, z)* with clock bias t_b is

$$\rho(x, y, z, t_b) = R(x, y, z) + ct_b \qquad [2]$$

Ignoring second and higher order terms, this can be approximated and linearised by a multi-variable, generalised Taylor Series expansion

$$\rho(x + \Delta x, y + \Delta y, z + \Delta z, t + \Delta t) \approx \rho(x, y, z, t) + \frac{\partial \rho}{\partial x}\Delta x + \frac{\partial \rho}{\partial y}\Delta y + \frac{\partial \rho}{\partial z}\Delta z + \frac{\partial \rho}{\partial t}\Delta t \qquad [3]$$

Let *(x, y, z)* and time t_b be estimates of receiver position and clock bias; and let Δx, Δy, Δz and Δt_b be errors in the current estimates. Better estimates are

$$x' = x + \Delta x \qquad y' = y + \Delta y \qquad z' = z + \Delta z \qquad t_b' = t_b + \Delta t_b$$

And equation 3 can be re-written

$$\rho(x', y', z', t_b') \approx \rho(x, y, z, t_b) + \frac{\partial \rho}{\partial x}\Delta x + \frac{\partial \rho}{\partial y}\Delta y + \frac{\partial \rho}{\partial z}\Delta z + \frac{\partial \rho}{\partial t}\Delta t_b \qquad [4]$$

Abbreviating *R(x, y, z)* to *R*, the partial derivates of ρ are

$$\frac{\partial \rho}{\partial x} = \frac{x - x_s}{R} \qquad \frac{\partial \rho}{\partial y} = \frac{y - y_s}{R} \qquad \frac{\partial \rho}{\partial z} = \frac{z - z_s}{R} \qquad \frac{\partial \rho}{\partial t} = c$$

Substituting into equation 4 gives

$$\rho(x', y', z', t_b') \approx \rho(x, y, z, t_b) + \frac{x - x_s}{R}\Delta x + \frac{y - y_s}{R}\Delta y + \frac{z - z_s}{R}\Delta z + c\Delta t_b \qquad [5]$$

The (notionally more accurate) left-hand-side of equation 5 is now replaced with an actual pseudo-range measurement made by the GPS receiver

$$ct_{RX} - ct_{TX} = \rho(x, y, z, t_b) + \frac{x - x_s}{R}\Delta x + \frac{y - y_s}{R}\Delta y + \frac{z - z_s}{R}\Delta z + c\Delta t_b \qquad [6]$$

This is a linear equation in four unknowns: Δx, Δy, Δz, and Δt_b.

Let error term $\Delta\rho$ be the difference between measured and estimated pseudo-range

$$\Delta\rho = ct_{RX} - ct_{TX} - \rho(x, y, z, t_b)$$

And re-write equation 6 in terms of $\Delta\rho$

$$\Delta\rho_s = \frac{x - x_s}{R_s}\Delta x + \frac{y - y_s}{R_s}\Delta y + \frac{z - z_s}{R_s}\Delta z + c\Delta t_b \qquad [7]$$

At least four range measurements are needed to solve for four unknowns. The simultaneous equations are conveniently written in matrix form

$$\begin{vmatrix} \Delta\rho_1 \\ \Delta\rho_2 \\ \Delta\rho_3 \\ \Delta\rho_4 \end{vmatrix} = \begin{vmatrix} \frac{x - x_1}{R_1} & \frac{y - y_1}{R_1} & \frac{z - z_1}{R_1} & c \\ \frac{x - x_2}{R_2} & \frac{y - y_2}{R_2} & \frac{z - z_2}{R_2} & c \\ \frac{x - x_3}{R_3} & \frac{y - y_3}{R_3} & \frac{z - z_3}{R_3} & c \\ \frac{x - x_4}{R_4} & \frac{y - y_4}{R_4} & \frac{z - z_4}{R_4} & c \end{vmatrix} \bullet \begin{vmatrix} \Delta x \\ \Delta y \\ \Delta z \\ \Delta t_b \end{vmatrix} \qquad [8]$$

Equation 8 can be expressed as

$$\Delta\boldsymbol{\rho} = \boldsymbol{H} \cdot \Delta\boldsymbol{u} \qquad [9]$$

In the four-satellite case, we simply multiply the pseudo-range errors by the matrix inverse of $\boldsymbol{H}$ to find the four unknowns

$$\Delta\boldsymbol{u} = \boldsymbol{H}^{-1} \cdot \Delta\boldsymbol{\rho} \qquad [10]$$

To solve more than four equations, we take the generalised or left inverse of $\boldsymbol{H}$

$$\Delta\boldsymbol{u} = (\boldsymbol{H}^T \cdot \boldsymbol{H})^{-1} \cdot \boldsymbol{H}^T \cdot \Delta\boldsymbol{\rho} \qquad [11]$$

$\boldsymbol{H}^T \cdot \boldsymbol{H}$ is always 4x4, so we still only need to compute a 4x4 inverse.

The influence of each satellite can be adjusted using weighted least squares

$$\Delta\boldsymbol{u} = (\boldsymbol{H}^T \cdot \boldsymbol{W} \cdot \boldsymbol{H})^{-1} \cdot \boldsymbol{H}^T \cdot \boldsymbol{W} \cdot \Delta\boldsymbol{\rho} \qquad [12]$$

The process is repeated until error magnitude $|\,\Delta x\ \Delta y\ \Delta z\,|$ is small. Typically, only five or six iterations are required. Zero starting values can be used for x, y, z and t_b.

Satellite positions should be transformed from ECEF to ECI co-ordinates inside the loop, since better estimates of signal flight times are available after each iteration.